

Indian Statistical Institute, Bangalore

B. Math (Hons.) First Year

Second Semester - Analysis II

Semester Exam

Maximum marks: 50

Date: 27th April 2026

Duration: 3 hours

Answer any 4 and each question carries 6 marks

1. Prove that any monotonic function on $[a, b]$ is Riemann-integrable.
2. If $f \in \mathcal{R}[0, 1]$ and $\epsilon > 0$, prove that there is a continuous function g on $[0, 1]$ such that $\int_0^1 |f - g| < \epsilon$.
3. Let $f \in \mathcal{R}[a, b]$ for all $b \geq a$. Prove that $\int_b^{\infty} f(x)dx$ converges if and only if for each $\epsilon > 0$, there exists a M such that $|\int_b^c f(x)dx| < \epsilon$ for all $c > b > M$.
4. Determine all functions f on $[0, \infty)$ such that the family $\{f_n \mid n \geq 1\}$ defined by $f_n(t) = f(nt)$ on $[0, 1]$ is equicontinuous.
5. Suppose series $\sum a_n x^n$ and $\sum b_n x^n$ converge in $(-R, R)$ and $E = \{x \in (-R, R) \mid \sum a_n x^n = \sum b_n x^n\}$ has a limit point. Prove that $a_n = b_n$ for all n .
6. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Prove that there is a subsequence (l_n) of integers and constants $c_{k,n}$ for $k = 0, 1, 2, \dots, l_n$ such that $g_n = \sum_{k=0}^{l_n} c_{k,n} f^k$ converges uniformly to $|f|$ on $[0, 1]$.

Answer any 2 and each question carries 13 marks

1. (a) Let $F: [a, b] \rightarrow \mathbb{R}$ be the anti-derivative of a function $f: [a, b] \rightarrow \mathbb{R}$. If $f \in \mathcal{R}[a, b]$, prove that $\int_a^b f(x)dx = F(b) - F(a)$ (Marks 6).
(b) Let $f: [0, 1] \rightarrow (0, \infty)$ be a continuous function and $M_n = (\int_0^1 f^n)^{\frac{1}{n}}$. Find $\lim_{n \rightarrow \infty} M_n$ and justify your answer.
2. (a) Let (f_n) be a pointwise bounded sequence of functions on a countable set E . Prove that there is a subsequence (f_{k_n}) that converges pointwise on E .
(b) Let $f_n \in \mathcal{R}[0, 1]$ be uniformly bounded. Define $F_n(x) = \int_0^x f_n(t)dt$ for all $x \in [0, 1]$. Prove that (F_n) has a uniformly convergent subsequence (Marks 6).
3. (a) Determine the convergence or divergence of the series using the integral test:
(i) $\sum_{n=1}^{\infty} n e^{-n^2}$; (ii) $\sum_{n=2}^{\infty} \frac{1}{n \log(n)}$ (Marks 6).
(b) Let $\sum_{n=0}^{\infty} c_n$ converge. Define $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for $x \in (-1, 1)$. Prove that $\lim_{x \rightarrow 1} f(x) = \sum_{n=0}^{\infty} c_n$.